

Graph Theoretic Construction of Discrete Groups over p -adic Fields

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1. Introduction

Let K be a finite extension of \mathbb{Q}_p and Γ a finitely generated discrete subgroup in $\mathrm{PGL}(2, K)$. It is well-known (e.g. [Her80, §1]) that such a group Γ contains a free normal subgroup of finite index (so called *Schottky group*). Hence it gives, through Mumford uniformization of an analytic curve, a Galois covering, possibly with ramifications, over a projective curve X_Γ with the Galois group Γ . Like that Mumford uniformization is linked with the corresponding geometry of (a subtree of) Bruhat-Tits tree, the situation as above is well described by the action of Γ on such trees (cf. [vdP97], [CKK99], and Proposition 2.13 below). For instance, the number of branch points and the branching degrees can be calculated only by looking at the corresponding trees. In slightly more precise terms, the discrete group Γ gives rise to a certain subtree \mathcal{T}_Γ^* in the Bruhat-Tits tree acted on by Γ , and the quotient graph $T_\Gamma^* = \Gamma \backslash \mathcal{T}_\Gamma^*$ can be decorated to a *graph of groups* $(T_\Gamma^*, \Gamma_\bullet)$, that is, a graph on which finite groups are attached to vertices and edges in a compatible way. There are several nice aspects in it; for instance, the genus of X_Γ is the first Betti number of T_Γ^* , ends of T_Γ^* are in bijection with branch points which preserves the decomposition groups, Γ is isomorphic to the direct-limit group (essentially by amalgams) associated to it, etc.

In this paper we will discuss the converse of the above procedure in case X_Γ is a rational curve. More precisely, we will answer to the following

Question. *Given an abstract tree of groups (T, G_\bullet) , when can one find a finitely generated discrete subgroup Γ in $\mathrm{PGL}(2, K)$ over some K , isomorphic to the amalgam group associated to (T, G_\bullet) , such that the tree of groups $(T_\Gamma^*, \Gamma_\bullet)$ “essentially” coincides with the original (T, G_\bullet) , i.e., roughly speaking, when is (T, G_\bullet) realizable in the above context?*

The meaning of “essentially” is that these trees of groups are the same modulo finite subtrees which are, so to speak, futile parts both topologically and group theoretically; this can be precisely stated by the notion of *contraction* (cf. [CKK99, Prop. 1] and Definition 2.15 below).

To answer the question, we will introduce the so-called $*$ -*admissibility* for such abstract trees of groups; roughly speaking, a tree of groups is $*$ -admissible if and only if it has nice compatible embeddings into the Bruhat-Tits tree and $\mathrm{PGL}(2, K)$ satisfying a certain local condition, local at each vertex. What we will prove is that this actually gives the necessary and sufficient condition for the realizability.

Moreover, this condition leads to a practical way of constructing discrete subgroups. For instance, suppose we are interested in classifying all the possible such groups with given number of branch points and branching degrees. Our method basically reduces the problem into a combinatorial business which is often easy in principle. In §4, we will exhibit two examples of such constructions. Our method actually has more applications (e.g. p -adic analogue of triangle groups, quadrangle groups, etc.), some of which will be discussed elsewhere (cf. [Kat00]).

It should be noted that, as one finds in the first example in §4, our method of construction may be viewed as a paraphrase of the known method by isometric circles (e.g. [Her80]). But the advantage of ours is its clear link with the tree which enable us to construct more complicated groups. In the second example in §4 we will construct a series of diadic triangle groups, which actually answers affirmatively to Yves André's expectation that there should be infinitely many p -adic *non-arithmetic* triangle groups (cf. [And98, 9.4]).

The plan of this paper is as follows: In the next section (§2) we will collect basic notions such as Bruhat-Tits tree, subtrees associated to discrete groups, and trees of groups, etc. (This section contains nothing new.) The $*$ -admissibility will be introduced in §3, where our main theorem will be proved. The final section §4 exhibit applications.

Notation and conventions. Throughout this paper K denotes a finite extension of \mathbb{Q}_p , \mathcal{O}_K the integer ring, and π a prime element in \mathcal{O}_K . We write $[K:\mathbb{Q}_p] = ef$, where e is the ramification degree and $q = p^f$ is the the number of elements in the residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$. We denote by $\nu: K^\times \rightarrow \mathbb{Z}$ the normalized (i.e., $\nu(\pi) = 1$) valuation.

2. Trees and groups

This section contains nothing essentially new. The notions and statements in this section can be found in the references listed in the end of this paper. Proofs are put for the reader's convenience, and are often sketchy.

2.1. Bruhat-Tits tree. First we recall the basic properties of Bruhat-Tits tree \mathcal{T}_K attached to $\mathrm{PGL}(2, K)$. It is the tree whose vertices are similarity classes of \mathcal{O}_K -lattices in K^2 , and two vertices are connected by an edge if the corresponding quotient module has length one. There is a canonical action by $\mathrm{PGL}(2, K)$ on \mathcal{T}_K . For a vertex v , which is the similarity class of $M \subset K^2$, edges emanating from v are in canonical bijection with the lines in $M/\pi M \cong k^2$, i.e., k -rational points of $\mathrm{Proj} \mathrm{Sym}_k(M/\pi M) \cong \mathbb{P}_k^1$:

$$(2.1.1) \quad \{\text{Edges } \sigma \text{ in } \mathcal{T}_K \text{ emanating from } v\} \longleftrightarrow \mathbb{P}^1(k).$$

The set of ends (i.e. equivalence classes of half-lines different by a finite segment) are canonically identified with K -rational points of \mathbb{P}_K^1 , since they are “limits” of sequences of lattices with length one successive quotients.

$$(2.1.2) \quad \{\text{Ends in } \mathcal{T}_K\} \longleftrightarrow \mathbb{P}^1(K).$$

Note that this bijection is equivariant with the action by $\mathrm{PGL}(2, K)$.

2.2. Notation. For an abstract tree T we denote by $\text{Vert}(T)$ (resp. $\text{Edge}(T)$, $\text{Ends}(T)$) the set of all vertices (resp. unoriented edges, ends). The notation $v \vdash \sigma$ for $v \in \text{Vert}(T)$ and $\sigma \in \text{Edge}(T)$ means that σ emanates from v . For a vertex $v \in \text{Vert}(T)$ we denote by $\text{Star}_v(T)$ the set of edges in $\text{Edge}(T)$ emanating from v . For two vertices v_0 and v_1 , we denote by $[v_0, v_1]$ the geodesic path connecting them. For $\varepsilon_0, \varepsilon_1 \in \text{Ends}(T)$ and $v \in \text{Vert}(T)$, the unique straight-line (resp. half-line) connecting ε_0 and ε_1 (resp. v and ε_0) is denoted by $]\varepsilon_0, \varepsilon_1[$ (resp. $[v, \varepsilon_0[$). The geometric realization $|T|$ is metrized so that the path $[v_0, v_1]$ ($v_0, v_1 \in \text{Vert}(T)$) is of length equal to the number of edges in it. The metric function is denoted by $d_T(\cdot, \cdot)$, or simply by $d(\cdot, \cdot)$. If $T \subseteq \mathcal{T}_K$, then we always regard the set $\text{Ends}(T)$ as a subset of $\mathbb{P}^1(K)$ by (2.1.2).

2.3. Lemma. *Let \mathcal{T} be a subtree of \mathcal{T}_K . Then the set of ends of \mathcal{T} , regarded as a subset in $\mathbb{P}^1(K)$, is a closed (hence compact) set.*

Proof. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a set of ends of \mathcal{T} which converges, as points in $\mathbb{P}^1(K)$, to a point ε . What to prove is that ε is contained in $\text{Ends}(\mathcal{T})$. For each n , let u_n be the vertex of \mathcal{T} determined by $]\varepsilon_{n-1}, \varepsilon_n[\cap]\varepsilon_n, \varepsilon_{n+1}[=]\varepsilon_n, u_n[$. Then the union of all the segments $[u_{n-1}, u_n]$ in \mathcal{T} contains a half-line ℓ pointing to the end ε . \square

2.4. Tree from a compact set. Next we recall the definition of trees from compact sets ([CKK99, (2.4)]): Let \mathcal{L} be a compact subset of $\mathbb{P}_K^{1,\text{an}}$. We assume that every point in \mathcal{L} is at most K -valued. The *tree generated by \mathcal{L}* , denoted by $\mathcal{T}(\mathcal{L})$, is the minimal subtree in \mathcal{T}_K having \mathcal{L} as the set of ends; it is an empty tree if \mathcal{L} consists of less than 2 points. This notion depends on the base field K , but differs only by subdivision. Note also that the tree $\mathcal{T}(\mathcal{L})$ in general differs from the one by Gerritzen-van der Put [GvP80, I.§2]; for instance, the tree $\mathcal{T}^{\text{GvdP}}(\mathcal{L})$ by them is a finite tree for \mathcal{L} being finite, whereas ours are not. In fact, we have the following criterion:

(2.4.1) *The tree $\mathcal{T}^{\text{GvdP}}(\mathcal{L})$ coincides with $\mathcal{T}(\mathcal{L})$ if and only if $\text{Ends}(\mathcal{T}^{\text{GvdP}}(\mathcal{L})) = \mathcal{L}$.*

This can be easily seen by the fact that $\mathcal{T}(\mathcal{L})$ is the minimal subtree containing all the apartments $]z, w[$ for $z, w \in \mathcal{L}$ ($z \neq w$).

2.5. Elements in a discrete subgroup. The following facts are well-known, but are inserted herein for the reader's convenience: An element $\gamma \in \text{PGL}(2, K)$ is said to be *parabolic* (resp. *elliptic*, resp. *hyperbolic*) if it has only one eigenvalue (resp. two distinct eigenvalues with equal valuations, resp. two distinct eigenvalues with different valuations). Let Γ be a discrete subgroup of $\text{PGL}(2, K)$. Then:

(2.5.1) *There exists no parabolic element in Γ other than 1.*

(2.5.2) *An element $\gamma \in \Gamma$ is of finite order if and only if it is elliptic.*

Suppose two elements θ and χ have exactly one common fixed point $\infty \in \mathbb{P}^1(K)$. We may assume $\theta = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and $\chi = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $b \neq 0$. Then it is easy to see that $\theta\chi\theta^{-1}\chi^{-1}$ is a parabolic element. Hence:

(2.5.3) *No two elements in Γ have exactly one common fixed point in \mathbb{P}_K^1 .*

2.6. Lemma. *Let Γ be a subgroup in $\text{PGL}(2, K)$ acting on a subtree \mathcal{T} of \mathcal{T}_K .*

- (1) If Γ is discrete, then for each $v \in \mathcal{T}$ the stabilizer of v is a finite group.
(2) Conversely, if the stabilizer of at least one vertex v is finite, then Γ is discrete.

Proof. (1) is well-known (the stabilizer in $\mathrm{PGL}(2, K)$ of a vertex is an open compact subgroup.) Suppose that there is a sequence $\{\gamma_i\} \subset \Gamma$ converging to 1, then, except for finitely many γ_i 's, they are contained in the stabilizer of v , since the stabilizer of a vertex in $\mathrm{PGL}(2, K)$ is an open neighborhood of 1. \square

2.7. Trees from a discrete group. Let Γ be a finitely generated discrete subgroup in $\mathrm{PGL}(2, K)$. We may assume, replacing K by a finite extension if necessary, that every element ($\neq 1$) in Γ has at most K -valued fixed points in \mathbb{P}_K^1 (cf. [GvP80, I.3.1 (1)]). Then Γ acts on \mathcal{T}_K without inversion. Let

$$\begin{aligned}\mathcal{L}_\Gamma &= \text{the set of limit points of } \Gamma, \\ \mathcal{F}_\Gamma &= \text{the set of fixed points of elements } (\neq 1) \text{ in } \Gamma.\end{aligned}$$

These are subsets in $\mathbb{P}_K^{1,\text{an}}$ satisfying $\mathcal{L}_\Gamma \subseteq \overline{\mathcal{F}}_\Gamma$, where $\overline{\cdot}$ denotes the topological closure. These sets are, in general, not equal, and the difference $\overline{\mathcal{F}}_\Gamma - \mathcal{L}_\Gamma$ is a discrete set, consisting of fixed points of elliptic elements in Γ . Now define:

$$\mathcal{T}_\Gamma = \mathcal{T}(\mathcal{L}_\Gamma) \quad \text{and} \quad \mathcal{T}_\Gamma^* = \mathcal{T}(\overline{\mathcal{F}}_\Gamma).$$

Clearly, we have $\mathcal{T}_\Gamma \subseteq \mathcal{T}_\Gamma^*$. It is also clear that, for an inclusion $\Gamma_1 \subseteq \Gamma_2$ of finitely generated discrete subgroups, we have inclusions of trees $\mathcal{T}_{\Gamma_1} \subseteq \mathcal{T}_{\Gamma_2}$ and $\mathcal{T}_{\Gamma_1}^* \subseteq \mathcal{T}_{\Gamma_2}^*$. The trees \mathcal{T}_Γ and \mathcal{T}_Γ^* admit canonically an action by Γ without inversion.

2.8. Examples. (1) If Γ is a finite subgroup, then \mathcal{T}_Γ is empty. The notion of the other tree \mathcal{T}_Γ^* fits in with the following concept: For an elliptic element $\gamma \in \mathrm{PGL}(2, K)$ with the fixed points $z, w \in \mathbb{P}^1(K)$, we set

$$M(\gamma) =]z, w[,$$

and call it the *mirror* of γ (this definition of mirror slightly differs from that in [CKK99, (2.3)]; see Lemma 2.10 below). Then the tree \mathcal{T}_Γ^* is the minimal one which contains all the mirrors of elements ($\neq 1$) in Γ .

(2) If Γ is a free subgroup (i.e., so-called, *Schottky group*), then the trees \mathcal{T}_Γ and \mathcal{T}_Γ^* coincide with each other, and with the Gerritzen-van der Put tree $\mathcal{T}_\Gamma^{\mathrm{GvdP}}$ ([GvP80, I.2.6]), originally introduced by Mumford ([Mum72]); indeed, in this case, it is well-known that the set of ends of the latter tree recovers the set of limit points (cf. [Mum72, (1.19)]).

(3) In general, we have $\mathcal{T}_\Gamma^{\mathrm{GvdP}}(\mathcal{L}_\Gamma) = \mathcal{T}_\Gamma$ (this follows easily from [GvP80, I.3.1 (1)]), and the other tree \mathcal{T}_Γ^* is the minimal one containing \mathcal{T}_Γ and all the mirrors of elliptic elements in Γ .

2.9. Remark. (1) The idea of the terminology “mirror” stems from the analogy to reflection mirrors in the theory of reflection groups. In fact, any elliptic element fixes its mirror pointwise, and “rotates” the other parts (cf. Lemma 2.10).

(2) Let $\gamma, \theta \in \Gamma$ be elliptic elements. Then the mirrors $M(\gamma)$ and $M(\theta)$ shares an end (i.e., $M(\gamma) \cap M(\theta)$ contains a half-line) if and only if $\langle \gamma, \theta \rangle$ is a cyclic group.

This follows easily from (2.5.3). In particular, mirrors are in bijection with maximal finite cyclic subgroups in Γ .

2.10. Lemma. *Let n be the order of γ , and set $G = \langle \gamma \rangle$.*

(1) *Let $v_0 \in M(\gamma)$. If $(n, p) = 1$, then G acts freely on the $q - 1$ vertices adjacent to v_0 not lying on $M(\gamma)$, where q is the number of elements in the residue field k .*

(2) *Suppose $n = p^r$ for $r \geq 1$, and set $s = \nu(\zeta_{p^r} - 1)$, where ζ_{p^r} is a primitive p^r -th root of unity, and ν is the normalized (i.e. $\nu(\pi) = 1$) valuation. Then a vertex $v \in \mathcal{T}_K$ is fixed by G if and only if $0 \leq d(v, M(\gamma)) \leq s$.*

Proof. We may assume that $\gamma: z \mapsto \zeta_n z$, where z is the inhomogeneous coordinate. (1) follows from the fact that the adjacent vertices are in canonical one-to-one correspondence with points in $\mathbb{P}^1(k)$. (2) is due to an easy calculation collaborated with the following fact: Let $v_0 = [\mathcal{O}_K X_0 + \mathcal{O}_K X_1]$ and $v_1 = [\mathcal{O}_K(X_0 + u_0 X_1) + \mathcal{O}_K \pi X_1]$ with $u_0 \not\equiv 0 \pmod{\pi}$. If v is a vertex such that the path $[v_0, v]$ contains v_1 , then $v = [\mathcal{O}_K(X_0 + (\sum_{i=0}^{d-1} u_i \pi^i) X_1) + \mathcal{O}_K \pi^d X_1]$, where $d = d(v, v_0)$. \square

2.11. Proposition. *For a finitely generated discrete subgroup Γ in $\mathrm{PGL}(2, K)$, the tree \mathcal{T}_Γ is minimal among the subtrees in \mathcal{T}_K acted on by Γ .*

Proof. If Γ does not contain a hyperbolic element, then \mathcal{T}_Γ is empty and the proposition is vacuous. Let \mathcal{T} be a subtree in \mathcal{T}_K acted on by Γ . It is well-known that the set of limit points \mathcal{L}_Γ is the topological closure of the set of fixed points of hyperbolic elements. By Lemma 2.3, it suffices to show that, for any hyperbolic element $\gamma \in \Gamma$, the tree \mathcal{T} contains the apartment connecting the fixed points of γ . Let $v \in \mathrm{Vert}(\mathcal{T})$. Since γ does not fix any vertex in \mathcal{T}_K , the $v_n = \gamma^n v$ for $n \in \mathbb{Z}$ are all distinct. For each n , let u_n be the vertex determined by $[v_{n-1}, v_n] \cap [v_n, v_{n+1}] = [v_n, u_n]$. The vertices u_n are also all distinct. Then $\gamma[u_{n-1}, u_n] = [u_n, u_{n+1}]$, and hence γ fixes two ends of the apartment $\bigcup_{n \in \mathbb{Z}} [u_n, u_{n+1}]$ in \mathcal{T} . \square

2.12. Quotient graphs. We denote by T_Γ and T_Γ^* the quotient graph of \mathcal{T}_Γ and \mathcal{T}_Γ^* , respectively, by Γ . The quotient maps of these trees are, by slight abuse of notation, both denoted by ϱ_Γ . Let $\Omega_\Gamma = \mathbb{P}_K^{1,\mathrm{an}} - \mathcal{L}_\Gamma$, the corresponding analytic domain, and $\varpi_\Gamma: \Omega_\Gamma \rightarrow \Gamma \backslash \Omega_\Gamma$ the quotient map. It is well-known that the graph T_Γ is finite, and that the analytic space $\Gamma \backslash \Omega_\Gamma$ is the analytification of a non-singular projective curve. Ramification points of ϖ_Γ are fixed points of elliptic elements, or equivalently, points in $\overline{\mathcal{F}}_\Gamma - \mathcal{L}_\Gamma$. This leads to the following statement (cf. [vdP97]):

2.13. Proposition. *There exist canonical bijections, compatible with the quotient maps,*

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{Ramification points in} \\ \text{Ω_Γ of the map ϖ_Γ} \end{array} \right\} & \longleftrightarrow & \mathrm{Ends}(T_\Gamma^*) - \mathrm{Ends}(T_\Gamma) \\ \downarrow & & \downarrow \\ \left\{ \begin{array}{l} \text{Branch points in $\Gamma \backslash \Omega_\Gamma$} \end{array} \right\} & \longleftrightarrow & \mathrm{Ends}(T_\Gamma^*). \end{array}$$

Moreover, the decomposition group of a ramification point coincides with the stabilizer of the corresponding end. \square

2.14. Stabilizers and tree of groups. For $v \in \text{Vert}(\mathcal{T}_K)$ (resp. $\sigma \in \text{Edge}(\mathcal{T}_K)$) we denote by Γ_v (resp. Γ_σ) the stabilizer in Γ of v (resp. σ with orientation). These are finite groups, for Γ is discrete. Now we assume that the quotient graph T_Γ^* are trees. Then by [Ser80, I.4.1, Prop. 17], there exists a section $\iota_\Gamma: T_\Gamma^* \hookrightarrow \mathcal{T}_\Gamma^*$ of the quotient map ϱ_Γ . Such a section gives rise to the so-called *tree of groups* $(T_\Gamma^*, \Gamma_\bullet)$ ([Ser80, I.4.4, Def. 8]) by attaching the stabilizers Γ_v (resp. Γ_σ) to vertices $v \in \text{Vert}(\iota_\Gamma(T_\Gamma^*))$ (resp. edges $\sigma \in \text{Edge}(\iota_\Gamma(T_\Gamma^*))$). By [Ser80, I.4.5, Thm. 10], we see that Γ is generated by the finite subgroups Γ_v for $v \in \text{Vert}(\iota_\Gamma(T_\Gamma^*))$, and is isomorphic to the associated amalgam product

$$\Gamma \xrightarrow{\sim} \lim_{\longrightarrow} (T_\Gamma^*, \Gamma_\bullet).$$

A similar isomorphy with $(T_\Gamma^*, \Gamma_\bullet)$ replaced by the finite subtree of groups $(T_\Gamma, \Gamma_\bullet)$ is also true by the same reasoning.

2.15. Definition (cf. [CKK99, Prop. 1]). Let (T, G_\bullet) be an abstract tree of groups, and $T' \subseteq T$ a subtree. Then the induced tree of groups (T', G_\bullet) is said to be a *contraction* of (T, G_\bullet) if the following conditions are satisfied:

- (1) $\text{Ends}(T') = \text{Ends}(T)$.
- (2) For every vertex v of $T - T'$ the stabilizers of vertices on the path from v to v' are ordered increasingly with respect to inclusion upon approaching T' , where v' is the vertex in T' nearest to v .

If (T', G_\bullet) is a contraction of (T, G_\bullet) , then, by (1), T differs from T' only by (possibly infinitely many) finite pieces, and (2) means that these pieces are inessential; in particular, the associated amalgams coincide with each other.

2.16. Lemma. Let $\mathcal{T}' \subseteq \mathcal{T} \subseteq \mathcal{T}_K$ be inclusions of trees into the Bruhat-Tits tree \mathcal{T}_K , and $\Gamma \subset \text{PGL}(2, K)$ a finitely generated discrete subgroup. Suppose that both \mathcal{T}' and \mathcal{T} are stable under the action of Γ . Let $T = \Gamma \backslash \mathcal{T}$ and $T' = \Gamma \backslash \mathcal{T}'$, and (T, Γ_\bullet) and (T', Γ_\bullet) the trees of groups induced by a section $\iota: T \hookrightarrow \mathcal{T}$. Suppose that the inclusion $T' \hookrightarrow T$ gives the bijection between the set of ends. Then (T', Γ_\bullet) is a contraction of (T, Γ_\bullet) .

Proof. This is a slight generalization of [CKK99, Proposition 1], and can be proven by the same argument as in [CKK99, (3.6)]. \square

3. Realization of tree of groups

3.1. As we saw in 2.14 any finitely generated discrete subgroup $\Gamma \subset \text{PGL}(2, K)$ such that T_Γ^* is a tree gives rise to a tree of groups $(T_\Gamma^*, \Gamma_\bullet)$, which recovers the abstract group isomorphic to Γ as, so to speak, the “fundamental group” of the data $(T_\Gamma^*, \Gamma_\bullet)$. Moreover the data $(T_\Gamma^*, \Gamma_\bullet)$ also recovers \mathcal{T}_Γ^* as an abstract tree (cf. [Ser80, I.4.5, Thm. 10]), which one can call the “universal covering” of $(T_\Gamma^*, \Gamma_\bullet)$. Now the natural question rises: Given an abstract tree of groups, when can one realize its fundamental group as a discrete subgroup in $\text{PGL}(2, K)$ and the universal covering as a subtree in \mathcal{T}_K ? The objective of this section is to answer this question.

3.2. Let (T, G_\bullet) be an abstract tree of groups, that is, an abstract tree T to

which finite groups G_v and G_σ for $v \in \text{Vert}(T)$ and $\sigma \in \text{Edge}(T)$ are attached; among these groups are injective homomorphisms $G_\sigma \hookrightarrow G_v$ for each pair (v, σ) with $v \vdash \sigma$. Suppose that we are given embeddings $G_v \hookrightarrow \text{PGL}(2, K)$ for any $v \in \text{Vert}(T)$ compatible with each $G_\sigma \hookrightarrow G_v$ for $v \vdash \sigma$. Such embeddings, provided that K is large enough, gives rise to subtrees $\mathcal{T}_{G_v}^*$ as in 2.7. Set

$$\tilde{\mathcal{T}}_{G_\bullet} = \text{the minimal subtree in } \mathcal{T}_K \text{ containing all } \mathcal{T}_{G_v}^*.$$

The set of ends in $\tilde{\mathcal{T}}_{G_\bullet}$ is, therefore, the union of the set of ends in $\mathcal{T}_{G_v}^*$ for $v \in \text{Vert}(T)$. This tree is labelled by groups (not necessarily finite) \tilde{G}_\bullet as follows: For a vertex $v \in \text{Vert}(\tilde{\mathcal{T}}_{G_\bullet})$ the group \tilde{G}_v is the subgroup in $\text{PGL}(2, K)$ generated by $(G_u)_v$ (the stabilizer at v by the action of G_u on \mathcal{T}_K) for all $u \in \text{Vert}(T)$; the definition of the group \tilde{G}_σ for $\sigma \in \text{Edge}(\tilde{\mathcal{T}}_{G_\bullet})$ is similar, which is just the intersection of \tilde{G}_v 's at the two extremities.

3.3. Definition. An *admissible embedding* of an abstract tree of groups (T, G_\bullet) is an embedding $\iota: T \hookrightarrow \mathcal{T}_K$ of trees together with embeddings $G_v \hookrightarrow \text{PGL}(2, K)$ for any $v \in \text{Vert}(T)$ compatible with each $G_\sigma \hookrightarrow G_v$ for any $v \vdash \sigma$ such that the following conditions are satisfied:

- (1) $\iota(T) \subset \tilde{\mathcal{T}}_{G_\bullet}$.
- (2) For any $v \in \text{Vert}(T)$ and $\gamma \in G_v$ ($\gamma \neq 1$), there exists $\delta \in \Gamma$ such that $M(\delta\gamma\delta^{-1}) \cap \iota(T)$ contains an edge, where Γ is the subgroup in $\text{PGL}(2, K)$ generated by all G_v for $v \in \text{Vert}(T)$.
- (3) $\tilde{G}_{\iota(v)} = G_v$ for any $v \in \text{Vert}(T)$.
- (4) $\tilde{G}_{\iota(\sigma)} = G_\sigma$ for any $\sigma \in \text{Edge}(T)$.
- (5) For any $v \in \text{Vert}(T)$, we have $\text{Star}_v(T) \cong G_v \backslash (G_v \cdot \text{Star}_{\iota(v)}(\tilde{\mathcal{T}}_{G_\bullet}))$ by the composite of ι followed by the projection.

The last condition means that T behaves locally like a fundamental domain at each vertex.

3.4. Definition. An abstract tree of groups (T, G_\bullet) is said to be **-admissible* if it has an admissible embedding and the associated amalgam $\lim_{\rightarrow}(T, G_\bullet)$ is finitely generated.

3.5. Lemma. If $\Gamma \in \text{PGL}(2, K)$ is a finitely generated discrete subgroup such that T_Γ^* is a tree, then $(T_\Gamma^*, \Gamma_\bullet)$ by a section $\iota_\Gamma: T_\Gamma^* \hookrightarrow \mathcal{T}_\Gamma^*$ is **-admissible*.

Proof. By 2.14, $\lim_{\rightarrow}(T, G_\bullet) \cong \Gamma$, and is finitely generated. Clearly, we have $\iota_\Gamma(T_\Gamma^*) \subset \tilde{\mathcal{T}}_{\Gamma_\bullet}$. (3.3.3) and (3.3.4) are obvious. Let $v \in \text{Vert}(T_\Gamma^*)$. Then $\text{Star}_v(T_\Gamma^*) \cong \Gamma_v \backslash \text{Star}_v(\mathcal{T}_\Gamma^*)$ obviously holds. But since $\iota_\Gamma(T_\Gamma^*) \subset \tilde{\mathcal{T}}_{\Gamma_\bullet} \subset \mathcal{T}_\Gamma^*$, we have (3.3.5). Finally, for $\gamma \in G_v$ with $\gamma \neq 1$, since T^* is a fundamental domain in \mathcal{T}_Γ^* , there exists $\delta \in \Gamma$ such that $M(\delta\gamma\delta^{-1}) \cap T$ is non-empty, containing a vertex w . Due to (3.3.5), of which we have proved the validity, one can further make a twist by $\chi \in G_w$ so that $M(\chi\delta\gamma\delta^{-1}\chi^{-1}) \cap T$ contains an edge. \square

Note that $(T_\Gamma, \Gamma_\bullet)$ is not **-admissible*, since it does not satisfy (3.3.5). What we are to show is that the converse of the above lemma in a certain sense:

3.6. Theorem. Let (T, G_\bullet) be a $*$ -admissible tree of groups and $\iota: T \hookrightarrow \tilde{\mathcal{T}}_{G_\bullet}$ with $\{G_v \hookrightarrow \mathrm{PGL}(2, K)\}_{v \in \mathrm{Vert}(T)}$ an admissible embedding. Let Γ be the subgroup in $\mathrm{PGL}(2, K)$ generated by all G_v for $v \in \mathrm{Vert}(T)$ and set

$$\mathcal{T}^* = \bigcup_{\gamma \in \Gamma} \gamma \cdot \iota(T)$$

in \mathcal{T}_K . Then:

- (1) The group Γ is a finitely generated discrete subgroup in $\mathrm{PGL}(2, K)$ isomorphic to $\lim_{\rightarrow} (T, G_\bullet)$.
- (2) The subset \mathcal{T}^* in \mathcal{T}_K is a tree and $\Gamma \backslash \mathcal{T}^* \cong T$.
- (3) The embedding ι gives a section $T \hookrightarrow \mathcal{T}^*$ by which the induced tree of groups (T, Γ_\bullet) equals to (T, G_\bullet) .

Moreover, if \mathcal{T}_Γ^* is the tree associated to Γ as in 2.7, then $\mathcal{T}_\Gamma^* \subseteq \mathcal{T}^*$, and the induced inclusion $\mathcal{T}_\Gamma^* \hookrightarrow T$ enjoys the following:

- (4) The induced inclusion $\mathrm{Ends}(\mathcal{T}_\Gamma^*) \hookrightarrow \mathrm{Ends}(T)$ is a bijection.
- (5) The section ι restricts to a section $\mathcal{T}_\Gamma^* \hookrightarrow \mathcal{T}_\Gamma^*$ by which the induced tree of groups $(\mathcal{T}_\Gamma^*, \Gamma_\bullet)$ is the restriction of $(T, \Gamma_\bullet) = (T, G_\bullet)$.
- (6) The tree of groups $(\mathcal{T}_\Gamma^*, \Gamma_\bullet)$ is a contraction of $(T, \Gamma_\bullet) = (T, G_\bullet)$.

3.7. To prove the theorem, we need several lemmas. In the sequel, we regard T as a subtree in \mathcal{T}_K by ι ; also, we can simply write G_v and G_σ instead of \tilde{G}_v and \tilde{G}_σ , respectively, because of (3.3.3) and (3.3.4). Trees are often regarded as metric spaces by geometric realization (cf. 2.2).

Let Γ be as in the theorem. Any element $\gamma \in \Gamma$ is expressed as $\gamma = \alpha_1 \cdots \alpha_m$ with $\alpha_i \in G_{v_i}$ for some $v_i \in \mathrm{Vert}(T)$ for $i = 1, \dots, m$. The length of γ is the minimal m in among all such expressions as above. Let $\Gamma^{(m)}$ be the set of all elements in Γ of length m . Obviously, $\Gamma^{(0)} = \{1\}$ and $\Gamma^{(1)} = \bigcup_{v \in \mathrm{Vert}(T)} G_v - \{1\}$.

3.8. Lemma. There is no half-line in T on which only trivial groups are attached to vertices and edges.

Proof. From the definition of the tree $\tilde{\mathcal{T}}_{G_\bullet}$, it follows that $\mathrm{Ends}(\tilde{\mathcal{T}}_{G_\bullet})$ is, regarded as a subset in $\mathbb{P}^1(K)$, the set of fixed points of elements in $\Gamma^{(1)}$. Then the lemma follows from (3.3.1), (3.3.3), and (3.3.4). (Note that if $G_u = 1$ then $\mathcal{T}_{G_u}^* = \emptyset$.) \square

3.9. Lemma. The subset \mathcal{T}^* in \mathcal{T}_K is arcwise connected, i.e. a subtree.

Proof. For any $\gamma = \alpha_1 \cdots \alpha_m \in \Gamma^{(m)}$, set $\gamma_i = \alpha_1 \cdots \alpha_i$ for $i = 1, \dots, m$ (set $\gamma_0 = 1$). Then $\gamma_i T \cap \gamma_{i+1} T = \gamma_i (T \cap \alpha_{i+1} T) \neq \emptyset$ for $i = 0, \dots, m-1$. Hence a point in γT can be connected by a path with a point in T . \square

3.10. Lemma. For $\gamma \in \Gamma^{(1)}$ and $v \in \mathrm{Vert}(T)$, $\gamma v \in \mathrm{Vert}(T)$ implies $\gamma v = v$.

Proof. Take $u \in \mathrm{Vert}(T)$ such that $\gamma \in G_u$. Let $w \in \mathrm{Vert}(T)$ be the vertex determined by $[u, v] \cap [u, \gamma v] = [u, w]$. Since $\gamma u = u$ we have $\gamma w = w$, i.e., $\gamma \in G_w$. If $\gamma v \neq v$, then the segments $[w, v]$ and $[w, \gamma v]$ in T contain edges, different from

each other, emanating from w which are in the same γ -orbit. But this contradicts (3.3.5). \square

3.11. Lemma. *Let $\gamma \in \Gamma^{(m)}$ with $m > 1$. Then $T \cap \gamma T = \emptyset$. Moreover, if $\gamma = \alpha_1 \cdots \alpha_m$ a minimal expression, and $v_1, v_2 \in \text{Vert}(T)$ with $\alpha_1 \in G_{v_1}$ and $\alpha_2 \in G_{v_2}$ are chosen so that $d(v_1, v_2)$ is minimal, then the geodesic path connecting T and γT contains $\alpha_1 v_2$.*

Proof. The proof is done by induction with respect to m . First we show the lemma in $m = 2$; $\gamma = \alpha_1 \alpha_2$. Since $\alpha_1 \notin G_{v_2}$, $\alpha_1 v_2 = \gamma v_2 \notin \text{Vert}(T)$ (due to Lemma 3.10). Now suppose $v \in T \cap \gamma T$. Then $[v, \alpha_1 v_2] = [v, \gamma v_2] \subseteq \gamma T$. Due to the minimality of $d(v_1, v_2)$, we have $[v_1, \alpha_1 v_2] \cap T = \{v_1\}$. Hence the geodesic path connecting γv_2 with the vertex v in T contains v_1 ; in particular, $v_1 \in T \cap \gamma T$. This means $\gamma^{-1} v_1 \in \text{Vert}(T)$, while $\gamma^{-1} v_1 \neq v_1$ (since $\gamma \notin \Gamma^{(1)}$). But $\gamma^{-1} v_1 = \alpha_2^{-1} v_1$ leads to contradiction to Lemma 3.10. Therefore, $T \cap \gamma T = \emptyset$.

Due to the minimality of $d(v_1, v_2)$, $[v_1, \alpha_1 v_2] \cap T = \{v_1\}$ and $[v_2, \alpha_2^{-1} v_1] \cap T = \{v_2\}$. This last equality gives $[\alpha_1 v_2, v_1] \cap \gamma T = \{\alpha_1 v_2\}$. Hence the segment $[v_1, \alpha_1 v_2]$ is the geodesic path connecting T and γT , which contains $\alpha_1 v_2$.

For $m > 2$, we set $\gamma' = \alpha_1 \cdots \alpha_{m-1}$. Take $v_m \in \text{Vert}(T)$ such that $\alpha_m \in G_{v_m}$. Then $\gamma v_m = \gamma' v_m \in \gamma' T \cap \gamma T$. Suppose $v \in T \cap \gamma T$. By induction, $[\gamma' v_m, v]$ contains $\alpha_1 v_2$. But this segment $[\gamma' v_m, v]$ is included in γT , which means $\gamma T \cap \alpha_1 T \neq \emptyset$. This contradics to the induction hypothesis, since $\gamma T \cap \alpha_1 T = \alpha_1 (\alpha_2 \cdots \alpha_m T \cap T)$. Hence $T \cap \gamma T = \emptyset$. Since the geodesic connecting γT and T contains that connecting $\gamma' T$ and T (since $\gamma' T \cap \gamma T \neq \emptyset$), in particular, it contains $\alpha_1 v_2$. \square

3.12. Corollary. *For $\gamma \in \Gamma$ ($\gamma \neq 1$), $T \cap \gamma T \neq \emptyset$ if and only if $\gamma \in \Gamma^{(1)}$.* \square

3.13. Corollary. *For $\gamma \in \Gamma$ and $v \in \text{Vert}(T)$, $\gamma v \in \text{Vert}(T)$ imples $\gamma \in G_v$.* \square

3.14. Corollary. *For $\gamma, \delta \in \Gamma$, and suppose $T \cap \gamma T \cap \gamma \delta T \neq \emptyset$. Then, for any $v \in T \cap \gamma T \cap \gamma \delta T$, we have $\gamma, \delta \in G_v$.*

Proof. By Lemma 3.11, γ , δ , and $\gamma \delta$ are in $\Gamma^{(1)} \cup \{1\}$. Since $\gamma^{-1} v \in \text{Vert}(T)$, $\gamma^{-1} v = v$ (by Lemma 3.10), which gives $\gamma \in G_v$. Similarly, we get $\delta \in G_v$. \square

By these corollaries and [Ser80, Appendix, pp. 30–31], we have:

3.15. Corollary. *Let F be the free group with basis X_α indexed by $\alpha \in \Gamma^{(1)} \cup \{1\}$, and $\varphi: F \rightarrow \Gamma$ the natural homomorphism $X_\alpha \mapsto \alpha$. Then $\text{Ker } \varphi$ is the normal subgroup generated by $X_\alpha X_\beta (X_{\alpha\beta})^{-1}$ for all (α, β) such that $\alpha, \beta \in G_v$ for some $v \in \text{Vert}(T)$.* \square

3.16. Corollary. *The natural homomorphism*

$$\lim_{\longrightarrow} (T, G_\bullet) \longrightarrow \Gamma$$

is an isomorphism.

Proof. It suffices to show that the kernel of the homomorphism $F \rightarrow \lim_{\longrightarrow} (T, G_\bullet)$ sending $X_\alpha \mapsto \alpha$ is the normal subgroup generated by $X_\alpha X_\beta (X_{\alpha\beta})^{-1}$ for $\alpha, \beta \in G_v \subset \lim_{\longrightarrow} (T, G_\bullet)$ with some $v \in \text{Vert}(T)$. But this is obvious from the definition of amalgams. \square

3.17. Proof of Theorem 3.6. Let X be the abstract tree (“universal covering” of (T, G_\bullet)) as in [Ser80, I.4.5 Theorem 9]. We first show that our tree \mathcal{T}^* and X are Γ -equivariantly isomorphic. To see this, it suffices to show that

$$\text{Vert}(\mathcal{T}^*) (= \Gamma \text{Vert}(T)) \cong \coprod_{v \in \text{Vert}(T)} G_T/G_v,$$

where $G_T = \lim_{\rightarrow}(T, G_\bullet)$, and that the similar equality holds also for the set of oriented edges. But these follow from Corollary 3.16 and Corollary 3.13.

Then it follows from [Ser80, I.4.5 Theorem 9] that T is a fundamental domain for \mathcal{T}^* modulo Γ , and the stabilizers Γ_v ($v \in \text{Vert}(T)$) and Γ_σ ($\sigma \in \text{Edge}(T)$) are equal to G_v and G_σ , respectively. In particular, by Lemma 2.6, Γ is discrete in $\text{PGL}(2, K)$. Therefore, (1) and (2) have been proved. The embedding ι obviously gives a section $T \rightarrow \mathcal{T}^*$, and hence, we have (3).

We proceed to the proof of (4)~(6). First we are going to show $\mathcal{T}_\Gamma^* \subseteq \mathcal{T}^*$. In view of Proposition 2.11, it suffices to show that the mirror of any elliptic element $\gamma \in \Gamma$ is contained in \mathcal{T}^* . Since γ is Γ -conjugate to an element in $\Gamma^{(1)}$ ([Ser80, I.1.3 Corollary 1]), by (3.3.2), we may assume that $\gamma \in G_v$ for a vertex $v \in \text{Vert}(T)$ and $M(\gamma) \cap T$ contains an edge. If $M(\gamma) \subseteq T$, there is nothing to prove. Otherwise, $M(\gamma) \cap T$ is either a half-line, a segment of finite length. If it is a half-line $\ell = [u, \varepsilon]$, then let us denote the other “half” by $\bar{\ell} = [u, \bar{\varepsilon}]$ ($M(\gamma) = \ell \cup \bar{\ell}$). Since ε and $\bar{\varepsilon}$ are in the same orbit by the action of G_u on $\mathbb{P}^1(K)$, we have $\delta \in G_u$ such that $\delta\ell = \bar{\ell}$. Hence $M(\gamma) \subseteq \mathcal{T}^*$. Suppose $M(\gamma) \cap T$ is a segment $[u, w]$. Let $[w_1, u]$ be in $M(\gamma)$ such that $d(w_1, u) = d(u, w)$ and $[w_1, u] \cap [u, w] = \{u\}$. Then in $\mathcal{T}_{G_u}^*$ these two segments are in the same G_u -orbit by the same reasoning as above for two half-lines in $M(\gamma)$ starting at u extending these two segments. We can find $\delta_1 \in G_u$ such that $\delta_1[u, w] = [w_1, u]$. We can do the same for $[w_1, w]$ looking at $w_1 = \delta_1(w)$ and $\delta_1 \mathcal{T}_{G_w}^* = \mathcal{T}_{G_{w_1}}^*$. We find $\delta_2 \in G_{w_1} \subset \Gamma$ such that $\delta_2[w_1, u] = [w_2, w_1]$ in $M(\gamma)$. Repeating this, we can inductively find w_n such that $[w_n, w]$ is in $M(\gamma)$ and $\delta_n \in \Gamma$ such that $\delta_n[w_n, w] = [w_{n+1}, w_n]$. These segments are in \mathcal{T}^* , and the union of them is a half-line starting at w contained in $M(\gamma)$. Similarly, we can find the other half in \mathcal{T}^* . Hence we have shown that all the mirrors of elliptic elements in Γ appear in \mathcal{T}^* , thereby $\mathcal{T}_\Gamma^* \subseteq \mathcal{T}^*$.

Next we claim that $\text{Ends}(\mathcal{T}^*)$ is, as a subset of $\mathbb{P}^1(K)$, equal to $\overline{\mathcal{F}}_\Gamma$ (cf. 2.7). It follows from $\mathcal{T}_\Gamma^* \subseteq \mathcal{T}^*$ that $\overline{\mathcal{F}}_\Gamma$ is contained in $\text{Ends}(\mathcal{T}^*)$. If there exists a half-line ℓ in \mathcal{T}^* pointing to $\varepsilon \notin \overline{\mathcal{F}}_\Gamma$, then, replaced by a subhalf-line if necessary, ℓ contains no vertex with a non-trivial stabilizer, and hence is mapped to a half-line in T on which the stabilizers of vertices and edges are all trivial groups; but this contradicts Lemma 3.8. Hence we get $\text{Ends}(\mathcal{T}^*) = \overline{\mathcal{F}}_\Gamma = \text{Ends}(\mathcal{T}_\Gamma^*)$, and we obtain the bijection in (4) by taking quotient by Γ .

Since the diagram of morphism of trees

$$\begin{array}{ccc} \mathcal{T}_\Gamma^* & \hookrightarrow & \mathcal{T}^* \\ \downarrow & & \downarrow \\ T_\Gamma^* & \hookrightarrow & T \end{array}$$

is cartesian (e.g. in the category of metric spaces), it follows that the section ι restricts to a section of T_Γ^* into \mathcal{T}_Γ^* . The other part of (5) is clear. (6) is due to Lemma 2.16. \square

To conclude this section, we insert herein a corollary to Theorem 3.6 useful for application. Let (T, G_\bullet) be a $*$ -admissible tree of groups. In view of Proposition 2.13 the ends of T are in bijection with branch points of $\Omega_\Gamma \rightarrow \Gamma \backslash \Omega_\Gamma \cong \mathbb{P}_K^1$. Hence $\text{Ends}(T)$ is a finite set. Let $\varepsilon \in \text{Ends}(T)$. Since $\lim_{\rightarrow} (T, G_\bullet)$ is finitely generated, we can find a half-line in T converging to ε such that the attached groups are ordered decreasingly with respect to inclusion upon approaching ε . We denote by G_ε the intersection of these groups, and call the stabilizer of ε . This is not a trivial group due to Lemma 3.8.

3.18. Corollary. *Let (T, G_\bullet) be a $*$ -admissible tree of groups. Then, for $\varepsilon \in \text{Ends}(T)$, the group G_ε is a finite cyclic group. Let $\text{Ends}(T) = \{\varepsilon_1, \dots, \varepsilon_n\}$, and $o(i)$ the order of G_{ε_i} for $i = 1, \dots, n$. Then there exists a finitely generated discrete subgroup Γ in $\text{PGL}(2, K)$ isomorphic to $\lim_{\rightarrow} (T, G_\bullet)$ such that $\Gamma \backslash \Omega_\Gamma \cong \mathbb{P}_K^1$ and the quotient map $\varpi_\Gamma: \Omega_\Gamma \rightarrow \Gamma \backslash \Omega_\Gamma$ branches over n points with branching degrees $o(\varepsilon_1), \dots, o(\varepsilon_n)$.*

Proof. All these are clear by the theorem and Proposition 2.13. \square

4. Examples

4.1. Free product (cf. [Her78, §11]). Let us begin with a simple example. Let (T, G_\bullet) be the tree of groups as drawn in Figure 1.

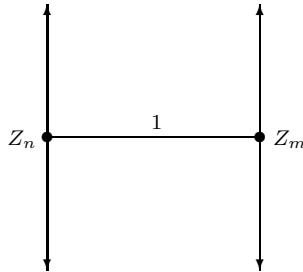


FIGURE 1

Here the four arrows stand for ends; to the left (resp. right) vertical line only the cyclic group Z_n of order n (resp. Z_m of order m) is attached, while the groups attached to the vertices on the horizontal line, except for its extremities, are all trivial groups.

Let us show that, provided that n and m are prime to p and that the length of the horizontal line is even, the (T, G_\bullet) is $*$ -admissible (the assumption on the length of the horizontal line is by no means essential, since one can always attain it by replacing K by a ramified quadratic extension): Let K be a finite extension of \mathbb{Q}_p containing ζ_n (resp. ζ_m), a primitive n -th (resp. m -th) root of unity, $\pi \in \mathcal{O}_K$ a prime element, and r the half of the length of the horizontal line. The embedding

$\iota: T \rightarrow \mathcal{T}_K$ is the one determined as follows: The left (resp. right) vertical line is mapped to the apartment $]0, \pi^r[$ (resp. $]\pi^{-r}, \infty[$). These apartments are disjoint and of distance $2r$; in fact, the segment connecting $v_0 = [\mathcal{O}_K\pi^r \oplus \mathcal{O}_K]$ and $v_1 = [\mathcal{O}_K \oplus \mathcal{O}_K\pi^r]$ is the geodesic path, which is the image of the horizontal segment in T . Let $\gamma, \delta \in \mathrm{PGL}(2, K)$ be defined by the fractional linear transformations:

$$\gamma(z) = \frac{\zeta_n \pi^r z}{(\zeta_n - 1)z + \pi^r}, \quad \delta(z) = \zeta_m z - (\zeta_m - 1)\pi^{-r},$$

where z is the inhomogenous coordinate. The element γ (resp. δ) is of order n (resp. m), and has 0 and π^r (resp. π^{-r} and ∞) as its fixed points. Let us embedd Z_n and Z_m in (T, G_\bullet) by fixing $Z_n \cong \langle \gamma \rangle$ and $Z_m \cong \langle \delta \rangle$. Then by Lemma 2.10, if n and m are prime to p , we see $\tilde{\mathcal{T}}_{G_\bullet} = \iota(T)$, and we can easily check that the above embedding is admissible. Hence by Theorem 3.6, the subgroup $\Gamma = \langle \gamma, \delta \rangle$ is discrete and isomorphic to the free product $Z_n * Z_m$. By Corollary 3.18, this Γ gives $\Omega_\Gamma \rightarrow \mathbb{P}_K^1$ branched over 4 points with branching degrees (n, n, m, m) .

If either n or m are not prime to p , then one can modify the groups on the horizontal line according to Lemma 2.10 (hence r should be large enough) to make it $*$ -admissible. Also in this case the associated group Γ is, provided r large enough, isomorphic to the free product $Z_n * Z_m$.

4.2. Triangle group. In this paragraph we assume $p = 2$. Let n be a positive odd number, and K a finite extension of \mathbb{Q}_2 containing ζ_n , and $\pi \in \mathcal{O}_K$ a prime. Let e be the ramification degree of K over \mathbb{Q}_2 . First consider the dihedral subgroup D_n generated by $\gamma, \chi \in \mathrm{PGL}(2, K)$ with

$$\gamma(z) = \zeta_n z, \quad \chi(z) = 1/z.$$

The fixed points of γ are 0 and ∞ , while those of χ are 1 and -1 . One sees easily that $M(\gamma) =]0, \infty[$ and $M(\chi) =]1, -1[$ are disjoint with distance e . Let v_0 (resp. v_1) be the vertex in \mathcal{T}_K which is the similarity class of the standard lattice $\mathcal{O}_K e_0 + \mathcal{O}_K e_1$ (resp. the lattice $\mathcal{O}_K(e_0 + e_1) + \mathcal{O}_K 2e_1$), where $e_0 = (1, 0)$ and $e_1 = (0, 1)$. Then the segment $[v_0, v_1]$ is the geodesic path connecting $M(\gamma)$ and $M(\chi)$. A fundamental domain $T_{D_n}^*$ for $\mathcal{T}_{D_n}^*$ modulo D_n is given by the union of (i) the half-line $[v_0, \infty[$, (ii) $M(\chi)$, and (iii) the segment $[v_0, v_1]$ (see Figure 2).

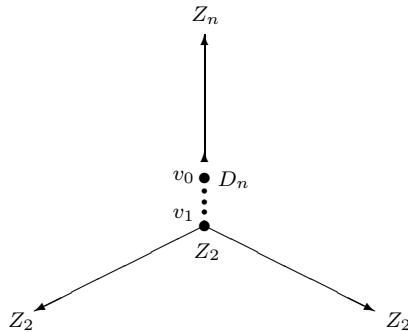


FIGURE 2: n : odd

The groups attached to $T_{D_n}^*$ are as follows: On $M(\chi)$ all vertices and edge are labelled by Z_2 . Vertices and edges on $[v_0, \infty[$, except for v_0 are labelled by Z_n , while v_0 is by the whole D_n . By Lemma 2.10, to vertices and edges in $[v_0, v_1]$ (denoted by the dotted segment), except for v_0 , the group Z_2 is attached. (Needless to say, they are subgroups of D_n).

Now consider $\theta \in \mathrm{PGL}(2, K)$ elliptic of order $2m$ with m odd such that $\theta^m = \chi$, i.e., θ has the same fixed points as χ . Then we can consider the (abstract) tree of groups as in Figure 2 with $M(\chi)$ replaced by $M(\theta)$, which amounts to replace all the Z_2 's on the lower straight-line in Figure 2 by Z_{2m} . It can be checked that the resulting tree of groups (T, G_\bullet) is $*$ -admissible by the obvious embeddings; that m is assumed to be odd guarantees (3.3.3) and (3.3.4) (on vertices and edges in $[v_0, v_1]$) due to Lemma 2.10. Although in this case the tree \tilde{T}_{G_\bullet} is bigger than T , validity of (3.3.5) follows from an argument similar to that in the previous example, and the fact that T came from the fundamental domain of D_4 .

It follows therefore that for any odd numbers n and m there exists a discrete subgroup Γ in $\mathrm{PGL}(2, K)$ (with K sufficiently large) isomorphic to $D_n *_{Z_2} Z_m$ such that the associated quotient map $\varpi: \Omega_\Gamma \rightarrow \Gamma \backslash \Omega_\Gamma \cong \mathbb{P}_K^1$ branches exactly above three points with branching degree $(n, 2m, 2m)$.

The assumption that m is odd is actually not essential; even if m is an even number, one can modify the groups attached to $[v_0, v_1]$, according to Lemma 2.10 so that the resulting tree of groups is $*$ -admissible.

4.3. Remark. (1) Note that, unless $p = 2$, the above construction does not work any more, since the fundamental domain of $T_{D_n}^*$ looks different; more precisely, if $p > 2$ then $M(\gamma)$ and $M(\chi)$ has non-empty intersection. Hence one cannot perform the replacement of groups as above.

(2) The resulting discrete group is a p -adic analogue of the Schwarzian triangle groups (cf. [And98, §9]). Our example gives an affirmative answer to Yves André's expectation (cf. [And98, 9.4]) that there will be infinitely many non-arithmetic p -adic triangle groups.

(3) Using the method as above, we can construct more triangle groups, not only in $p = 2$; by this, in particular, one can show that, for $p = 2, 3, 5$ there are infinitely many non-arithmetic triangle groups. Actually, one can also show that for $p > 5$ there is no triangle group constructed by this method. In order to discuss these, as the construction in 4.2 indicates, one has to describe the fundamental domains for finite groups of other types, i.e., D_n with n even, tetrahedral group, octahedral group, and icosahedral group. This will be done in [Kat00].

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